Betti numbers of monomial ideals and its application to combinatorics(Geometry of Toric Varieties and Convex Polytopes)

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Betti numbers of monomial ideals and its application to combinatorics

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序

これは、日比孝之氏 (大阪大学理学部) との共同研究についての概説です。詳しくは、[T–H$_2$]、[T–H$_3$] 、[T–H$_4$] を御覧になって下さい。

有限集合 $V = \{x_1, x_2, \ldots, x_v\}$ に対して、頂点集合 V 上の単体的複体 (simplicial complex) Δ を次の条件 (1)、(2) を満たす $\mathbf{2}^V$ の部分集合とする。但し、 $\mathbf{2}^V$ は V の部分集合全体からなる集合とする。
(1) $1 \leq i \leq v$ に対して、$\{x_i\} \in \Delta$。
(2) $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$。

$\sharp(\sigma)$ で 有限集合 σ の濃度を表すことにする。Δ の元 σを Δ の面 (face) という。特に、$\sharp(\sigma) = i+1$ のとき、 i-face という。$d = \max\{\sharp(\sigma) \mid \sigma \in \Delta\}$ とおき、 Δ の次元 (dimension) を $\dim \Delta = d-1$ で定義する。

$A = k[x_1, x_2, \ldots, x_v]$ を体 上の v変数多項式環とする。$V = \{x_1, x_2, \ldots, x_v\}$ 上の単体的複体 Δに対して A のイデアル I_Δ を次のように定義する。

$$I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_r} | 1 \leq i_1 < i_2 < \cdots < i_r \leq v,\ \{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta)$$

$k[\Delta] := A/I_\Delta$ をΔ の Stanley–Reisner 環という。

以後、A を $\deg x_i = 1$ として 次数付き環 $A = \bigoplus_{n\geq 0} A_n$ とみなす。すると、$k[\Delta]$ もまた、自然に A 上の次数付き加群 $k[\Delta] = \bigoplus_{n\geq 0}(k[\Delta])_n$ とみなせる。

$k[\Delta]$ の A 上の次数付き極小自由分解を

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0$$

とする。ここで、h を $k[\Delta]$ の ホモロジー次元 (homological dimension) といい、$h = \mathrm{hd}_A(k[\Delta])$ とあらわす。このとき $v-d \leq h \leq v$が 成り立つことが知られている。また、各β_{i_j} を $k[\Delta]$ のベッチ数 (Betti number) という。

Stanley-Reisner 環の Betti 数は、代数的な手法からだけでなく、トポロジーや組合せ論の手法を用いて、研究されてきている。著名な単体的複体の類に対して、その Stanley–Reisner 環の Betti 数を決定することは、興味深い問題であると思われる。本稿においては、まず、§1 で cyclic 多面体と stacked 多面体についてその定義と組合せ論的性質について述べる。それらが、組合せ論で重要なのは、頂点数と次元を固定したとき、それぞれが face の数の最大値と最小値を与えるからである。つづく§2 では、Stanley–Reisner 環の Betti 数を単体的複体のホモロジー群の言葉であらわす Hochster の公式について説明する。この公式が、Stanley–Reisner 環の Betti 数を組合せ論的手法を用いて研究できる基礎を与えている。以上の準備のもと、§3 と §4 で cyclic 多面体と stacked 多面体に付随する Stanley-Reisner 環の Betti 数を組合せ論的にあらわす公式を与える。そのことから、とくに、それらが基礎体 に依存しないことがわかる。最後の§5 では、極小自由分解の、comparability graph の連結度への応用について述べる。整数 i を $1 \leq i < v$ で 固定する。1 次元単体的複体 (グラフ)Δ に対して、 Δ から任意に (i-1) 個以下の頂点 (とそれに隣接する辺 (1-face)) を取り除いても、それが連結であるとき、Δ は i-連結 (i-connected) であると言う。単体的複体Δ の 1-骨格 (1-skeleton) とは

$$\Delta^{(1)} = \{\sigma \in \Delta\ ;\ \sharp(\sigma) \leq 2\}$$

で 定まる 1 次元部分複体である。半順序集合 Pの順序複体の 1-骨格を Pの comparability graph といい、 $\mathrm{Com}(P)$ とあらわす。そこで、極小自由分解の手法を用いて、ランク $d-1$ の非平面的分配束の comparability graph が d-連結であることを示す。

§1. Cyclic polytopes and stacked polytopes

In this section we briefly summarize the definition and basic facts of cyclic polytopes and stackd polytopes according to [Bil–Lee] and [Brø]. See those references for the detailed information.

Let $\mathbf{R}$ denote the set of real numbers. For any subset M of the d-dimensional Euclidean space $\mathbf{R}^d$, there is a smallest convex set containing M. We call this convex set the *convex hull* of M and denote it by $\mathrm{conv}M$. For $d \geq 2$ the *moment curve* in $\mathbf{R}^d$ is the curve parametrized by

$$t \longmapsto x(t) := (t, t^2, \cdots, t^d), \qquad t \in \mathbf{R}.$$

By a *cyclic polytope* $C(v,d)$, where $v \geq d+1$ and $d \geq 2$, we mean a polytope $\mathcal{P}$ of the form $\mathcal{P} = \mathrm{conv}\{x(t_1), \cdots, x(t_v)\}$, where $t_1, \cdots, t_v$ are distinct real numbers. It is well known that $C(v,d)$ is a simplicial d-polytope with the vertex set $\{x(t_1),\cdots,x(t_v)\}$, and its face lattice is independent of the particular values of t. Therefore its boundary complex is a simplicial complex and has the same combinatorial structure for any choices of vertices. We denote it by $\Delta(C(v,d))$.

Let $V=\{x_1,\cdots,x_v\}$ be the vertex set of $C(v,d)$. Let W be a proper subset of V. A subset X of W of the form $X=\{x_i,x_{i+1},\cdots,x_j\}$ is said to be a *contiguous subset* of W if $i>1, j<v, x_{i-1}\notin W$, and $x_{j+1}\notin W$. The set X is a *left end-set* of W if $i=1$ and $x_{j+1}\notin W$, and a *right end-set* of W if $j=v$ and $x_{i-1}\notin W$. We say that X is a *component* of W if X is a contiguous subset or an end-set of W. A subset X of W is said to be *even* (resp. *odd*) if the number of elements in X is even (resp. odd). The set W can be written uniquely in the form $W=Y_1\cup X_1\cup\cdots\cup X_n\cup Y_2$, where X_i, $1\le i\le n$, is a contiguous subset of W, and Y_i, $i=1,2$, is an end-set of W or an empty set. We quote two facts which are necessary later. We may abuse notation and call a subset W of V itself a face of $C(v,d)$ if conv$W$ is a face of $C(v,d)$.

(1.1) LEMMA ([Brø, Theorem 13.7]). *Let W be an m-element subset of V, where $m\le d$. Then W is an $(m-1)$-face of $C(v,d)$ if and only if the number of odd contiguous subsets of W is at most $d-m$.*

(1.2) LEMMA ([Brø, Corollary 13.8]). *Let m be an integer such that $1\le m\le[\frac{d}{2}]$. Then all m-element subsets of V are $(m-1)$-faces of $C(v,d)$.*

Now we define a stacked polytope inductively. Starting with a d-simplex, one can add new vertices by building shallow pyramids over facets to obtain a simplicial convex d-polytope with v vertices, called a *stacked polytope.*

UPPER BOUND AND LOWER BOUND THEOREM. *Let $\mathcal{P}$ be a d-dimensional simplicial polytope with v vetices. Let C* (resp. *S*) *be a d-dimensional cyclic (resp. stacked) polytope with v vetices. Then we have*

$$f_i(S)\le f_i(\mathcal{P})\le f_i(C),$$

where $f_i(\mathcal{P})$ stands for the number of i-facces of a polytope $\mathcal{P}$.

§2. Hochster's formula

Given a subset W of V, the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$.

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

Hochster's formula [Hoc, Theorem (5.1)] is that

$$\beta_{i_j} = \sum_{W \subset V,\ \sharp(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k).$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k).$$

Some combinatorial and algebraic applications of Hochster's formula have been studied. See, e.g., [Bac], [B–H$_1$], [B–H$_2$], [Mun], [H$_2$], [H$_3$], [H$_4$], [H$_5$], and [T–H$_1$].

§3. Betti numbers of Stanley–Reisner rings associated with cyclic polytopes

In this section we compute the Betti numbers of a minimal free resolution of the Stanley-Reisner ring $k[\Delta(C(v,d))]$ of the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$.

We fix a field k .

If the dimension d is even, a minimal free resolution of $k[\Delta]$ is pure and the Betti numbers can be easily computed from the Hilbert function of $k[\Delta]$.

(3.1) PROPOSITION ([Sch]). *Let Δ be the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$, where $d \geq 2$ is even. Then a minimal free resolution of $k[\Delta]$ over A is of the form:*

$$
\begin{aligned}
0 \longrightarrow A(-v) \longrightarrow A(-v+\frac{d}{2}+1)^{\beta_{v-d-1}} \longrightarrow \\
\cdots \longrightarrow A(-\frac{d}{2}-2)^{\beta_2} \longrightarrow A(-\frac{d}{2}-1)^{\beta_1} \longrightarrow A \longrightarrow k[\Delta] \longrightarrow 0,
\end{aligned}
$$

where for $1 \leq i \leq v-d-1$,

$$
\beta_i = \binom{v-\frac{d}{2}-1}{\frac{d}{2}+i}\binom{\frac{d}{2}+i-1}{\frac{d}{2}} + \binom{v-\frac{d}{2}-1}{i-1}\binom{v-\frac{d}{2}-i-1}{\frac{d}{2}}.
$$

Our formula on β_i in Proposition 3.1 is, in fact, a little bit different from the one in [Sch]. But it is easy to show that they are coincident.

If the dimension d is odd, the minimal free resolution of $k[\Delta]$ is not pure, and the situation is much more complicated.

Now we state the main theorem in this chapter.

(3.2) THEOREM. *Let Δ be the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$, where $d \geq 3$ is odd. Then a minimal free resolution of $k[\Delta]$ over A is of the form:*

$$
\begin{aligned}
0 \longrightarrow A(-v) \longrightarrow A\left(-v+\left[\frac{d}{2}\right]+2\right)^{b_{v-d-1}} \oplus A\left(-v+\left[\frac{d}{2}\right]+1\right)^{b_1} \longrightarrow \\
\cdots \longrightarrow A\left(-\left[\frac{d}{2}\right]-2\right)^{b_2} \oplus A\left(-\left[\frac{d}{2}\right]-3\right)^{b_{v-d-2}} \longrightarrow A\left(-\left[\frac{d}{2}\right]-1\right)^{b_1} \\
\oplus A\left(-\left[\frac{d}{2}\right]-2\right)^{b_{v-d-1}} \longrightarrow A \longrightarrow k[\Delta] \longrightarrow 0,
\end{aligned}
$$

where for $1 \leq i \leq v-d-1$,

$$
b_i = \binom{v-[\frac{d}{2}]-2}{[\frac{d}{2}]+i}\binom{[\frac{d}{2}]+i-1}{[\frac{d}{2}]}.
$$

Even if the geometric realization $|\Delta|$ of a simplicial complex Δ is a sphere, a Betti number of the Stanley-Reisner ring $k[\Delta]$ may depend on the base field k in general. See [T–H$_1$, Example 3.3]. But as for the boundary complexes of cyclic polytopes we have the following result:

(3.3) COROLLARY. *Let Δ be the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$, where $d \geq 2$. Then all the Betti numbers of the Stanley–Reisner ring $k[\Delta]$ are independent of the base field k.*

We show unimodality of the Betti number sequence $(\beta_0,\ \beta_1, \cdots,\ \beta_{v-d})$ of the Stanley-Reisner ring $k[\Delta(C(v,d))]$ associated with $C(v,d)$. Since this sequence is symmetric, i.e., $\beta_i = \beta_{v-d-i}$ for every $0 \leq i \leq v-d$, the unimodality means $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{[(v-d)/2]}$.

(3.4) COROLLARY. *Let Δ be the boundary complex $\Delta(C(v,d))$ of the cyclic polytope $C(v,d)$. Then, the Betti number sequence $(\beta_0(k[\Delta]),\ \beta_1(k[\Delta]), \cdots,\ \beta_{v-d}(k[\Delta]))$ of the Stanley-Reisner ring $k[\Delta]$ over A is unimodal.*

We prepare several lemmas to prove the theorem. We put $\Delta = \Delta(C(v,d))$ and $V = \{1, 2, \cdots, v\}$ for simplicity, and fix an odd integer $d \geq 3$.

(3.5) LEMMA. *If v is odd and $W = \{1,3,5,\cdots,v\}$, then*

$$\tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) = 0.$$

Proof. We have $\tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) \cong \tilde{H}_{[\frac{d}{2}]}(\Delta_{V-W}; k)$ by the Alexander duality theorem (see, e.g.,[Sta$_1$, p76]). Since $V - W = \{2,4,\ldots,v-1\}$, if σ is a subset of $V - W$ with $\sharp(\sigma) > [\frac{d}{2}]$, then σ does not belong to Δ by Lemma 1.1. Thus we have $\tilde{H}_{[\frac{d}{2}]}(\Delta_{V-W}; k) = 0$. Q.E.D.

(3.6) LEMMA. *If v is even and $W = \{1,3,5,\cdots,v-1\}$, then*

$$\tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) = 0.$$

Proof. All the maximal faces of Δ_W are of the form $\{1\} \cup \sigma$, where $1 \notin \sigma$, $\sigma \subset W$, $\sharp(\sigma) = [\frac{d}{2}]$. Thus Δ_W is a cone with apex $\{1\}$. Hence we have $\tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) = 0$. Q.E.D.

(3.7) LEMMA. *If v is even and $W = \{2,4,6,\cdots,v\}$, then*

$$\tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) = 0.$$

(3.8) LEMMA. *If v is odd and $W = \{2, 4, 6, \cdots, v-1\}$, then*

$$\dim_k \tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) = \binom{[\frac{v}{2}]-1}{[\frac{d}{2}]}.$$

Proof. Let

$$0 \to C_d \to C_{d-1} \to \cdots \to C_1 \to C_0 \to C_{-1} \to 0$$

be the augmented chain complex of the simplicial complex Δ_W over k. Then we have $C_{[\frac{d}{2}]} = 0$ and, for $j < [\frac{d}{2}]$, all the $(j+1)$-subsets of W form a basis of C_j as a vector space by Lemmas 1.1 and 1.2. Thus we have $\tilde{H}_j(\Delta_W; k) = 0$ for all $j < [\frac{d}{2}] - 1$. Hence, the Euler-Poincaré formula (see, e.g.,[Bru-Her, p223]) gives

$$\begin{aligned} & \dim_k \tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) \\ = \ & (-1)^{[\frac{d}{2}]}\left(\binom{[\frac{v}{2}]}{0} - \binom{[\frac{v}{2}]}{1} + \binom{[\frac{v}{2}]}{2} - \cdots + (-1)^{[\frac{d}{2}]}\binom{[\frac{v}{2}]}{[\frac{d}{2}]}\right) \\ = \ & \binom{[\frac{v}{2}]-1}{[\frac{d}{2}]}. \end{aligned}$$

Q.E.D.

(3.9) LEMMA. *Let W be a non-empty proper subset of V with a unique decomposition*

$$W = Y_1 \cup X_1 \cup X_2 \cup \ldots \cup X_n \cup Y_2$$

for some $n \geq 0$, where $X_i, 1 \leq i \leq n$, is a contiguous subset and $Y_i, i = 1, 2$, is an end-set or an empty set. Then

$$\dim_k \tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) = \begin{cases} \binom{n-1}{[\frac{d}{2}]}, & \textit{if } Y_1 = \emptyset \textit{ and } Y_2 = \emptyset, \\ 0, & \textit{otherwise,} \end{cases}$$

where we define $\binom{n-1}{[\frac{d}{2}]} = 0$ if $n - 1 < [\frac{d}{2}]$.

Proof. We prove the lemma by induction on the number v of vertices. First let $v = d + 1$. Then $C(v, d)$ is a d-simplex. Thus $\tilde{H}_{[\frac{d}{2}]-1}(\Delta_W; k) = 0$ for every subset $W \subset V$. Since $n \leq [\frac{v-1}{2}] = [\frac{d}{2}]$, the lemma holds.

Next let $v > d+1$. Let

$$V - W = X'_1 \cup X'_2 \cup \cdots \cup X'_{n+1}$$

be a unique decomposition, where $X'_i, 1 \le i \le n+1$, is a component of $V - W$. Suppose there exists X'_i $(1 \le i \le n+1)$ with $\sharp(X'_i) \ge 2$. Let j be an element of X'_i. Put $V' = V - \{j\}$. Note that $W \subset V'$. We consider the simplicial complex $\Delta' = \Delta(C(v-1,d))$ on the vertex set V'. Then we have $\Delta'_W = \Delta_W$ by Lemma 1.1. Thus we have $\tilde{H}_j(\Delta_W;k) = \tilde{H}_j(\Delta'_W;k)$. By the induction hypothesis, we are done in this case.

We put $X_0 := Y_1, X_{n+1} := Y_2$. Next suppose there exists X_i $(0 \le i \le n+1)$ with $\sharp(X_i) \ge 2$. Let j be an element of X_i. Put $V' = V - \{j\}$. We consider the simplicial complex $\Delta' := \Delta(C(v-1,d))$ on the vertex set V'. Then we have $\Delta'_{V-W} = \Delta_{V-W}$ by Lemma 1.1. By Alexander duality, we have

$$\tilde{H}_{[\frac{d}{2}]-1}(\Delta_W;k) \cong \tilde{H}_{[\frac{d}{2}]}(\Delta_{V-W};k) = \tilde{H}_{[\frac{d}{2}]}(\Delta'_{V-W};k) \cong \tilde{H}_{[\frac{d}{2}]-1}(\Delta'_W;k).$$

Thus we are done in this case.

In the remaining case we may assume $\sharp(X_i) = 1$ for $1 \le i \le n, \sharp(X'_i) = 1$ for $1 \le i \le n+1$ and $\sharp(Y_i) \le 1$ for $i = 1, 2$. But in this case we have the desired result by Lemmas 3.5, 3.6, 3.7, and 3.8. Q.E.D.

Proof of Theorem 3.2. Since $k[\Delta]$ is Gorenstein (see, e.g., [Bru–Her, Corollary 5.5.6]), we have $\mathrm{hd}_A k[\Delta] = v - d$. Let

$$0 \to F_{v-d} \to \cdots \to F_2 \to F_1 \to A \to k[\Delta] \to 0$$

be a minimal free resolution of $k[\Delta] = A/I_\Delta$ over A. By Lemma 1.2, we have $\min\{\alpha \in \mathbf{Z}; (I_\Delta)_\alpha \ne 0\} = [\frac{d}{2}] + 1$. Then F_1 has a direct summand of the form $A(-[\frac{d}{2}]-1)^{b_1}$ with $b_1 > 0$ and $F_i, 1 \le i \le v-d-1$, may have $A(-[\frac{d}{2}]-i)^{b_i}$ with $b_i \ge 0$ as a direct summand. We have $F_{v-d} = A(-v)$ and $F_i, 1 \le i \le v-d-1$, may have $A(-v+[\frac{d}{2}]+(v-d-i))^{b_{v-d-i}} = A(-[\frac{d}{2}]-i-1)^{b_{v-d-i}}$ as a direct summand by the self-duality of the minimal free resolution (see, e.g., [Sta$_1$, p59]). By [B–H$_2$, Proposition 1.1] we can easily check that other shiftings do not appear, since $k[\Delta]$ is Gorenstein. Thus we obtain the desired form of the minimal free resolution of $k[\Delta]$.

We now determine the graded Betti numbers $b_i, 1 \le i \le v-d-1$. By Hochster's formula we have

$$b_i = \beta_{i_{[\frac{d}{2}]+i}} = \sum_{W \subset V,\ \sharp(W) = [\frac{d}{2}]+i} \dim_k \tilde{H}_{[\frac{d}{2}]-1}(\Delta_W;k).$$

Let $c_i(n)$ denote the number of $([\frac{d}{2}]+i)$-subsets W of V such that W has a unique decomposition $W = X_1 \cup X_2 \cdots \cup X_n$ where $X_i, 1 \le i \le n$, is a contiguous subset of W. Then $c_i(n)$ is the number of positive integer solutions of the system of the equations

$$\begin{cases} x_1 + x_2 + \cdots + x_n = [\frac{d}{2}] + i \\ y_1 + y_2 + \cdots + y_{n+1} = v - [\frac{d}{2}] - i. \end{cases}$$

Thus we have

$$c_i(n) = \binom{[\frac{d}{2}]+i-1}{n-1}\binom{v-[\frac{d}{2}]-i-1}{n}.$$

By Lemma 3.9 and combinatorial identities in [Brø, Appendix 3] we have

$$\begin{aligned} b_i &= \sum_{n\ge 1} c_i(n)\binom{n-1}{[\frac{d}{2}]} \\ &= \sum_{n\ge 1}\binom{v-[\frac{d}{2}]-i-1}{n}\binom{[\frac{d}{2}]+i-1}{n-1}\binom{n-1}{[\frac{d}{2}]} \\ &= \sum_{n\ge 1}\binom{v-[\frac{d}{2}]-i-1}{n}\binom{[\frac{d}{2}]+i-1}{[\frac{d}{2}]}\binom{i-1}{n-[\frac{d}{2}]-1} \\ &= \binom{[\frac{d}{2}]+i-1}{[\frac{d}{2}]}\binom{v-[\frac{d}{2}]-2}{i+[\frac{d}{2}]}. \end{aligned}$$

Q.E.D.

§4. Betti numbers of Stanley–Reisner rings associated with stacked polytopes

Our main result in this chapter is to present a combinatorial formula for the computation of the Betti numbers of the Stanley–Reisner ring associated with the boundary complex of a stacked d-polytope $P(v,d)$ with v vertices.

(4.1) THEOREM. *Fix $v > d \ge 3$. Let $P(v,d)$ be a stacked d-polytope with v vertices and $\Delta(P(v,d))$ its boundary complex. Then, a minimal free resolution of the Stanley–Reisner ring $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ over A is of the form*

$$0 \longrightarrow A(-v) \longrightarrow A(-v+d)^{b_{v-d-1}} \bigoplus A(-v+2)^{b_1}$$

$$\longrightarrow A(-v+d+1)^{b_{v-d-2}} \bigoplus A(-v+3)^{b_2} \longrightarrow \cdots$$
$$\longrightarrow A(-3)^{b_2} \bigoplus A(-d-1)^{b_{v-d-2}}$$
$$\longrightarrow A(-2)^{b_1} \bigoplus A(-d)^{b_{v-d-1}} \longrightarrow A \longrightarrow k[\Delta(P(v,d))] \longrightarrow 0,$$

where

$$b_i = i\binom{v-d}{i+1}$$

for each $1 \leq i \leq v-d-1$.

When $d = 2$, $\Delta(P(v,2))$ is the cycle C_v with v vertices. A minimal free resolution of $k[C_v] = A/I_{C_v}$ over A is a *pure resolution*, which is discussed in, e.g., [B–H$_1$] and [B–H$_2$].

(4.2) COROLLARY. *Every Betti number of* $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ *over* A *is independent of the base field* k *and of the combinatorial type of* $P(v,d)$.

(4.3) COROLLARY. *Fix* $v > d \geq 3$. *Let* $P = P(v,d)$ *a stacked* d*-polytope with* v *vertices and* $\Delta = \Delta(P)$ *its boundary complex. Then, the Betti number sequence* $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ *of* $k[\Delta] = A/I_\Delta$ *over* A *is unimodal.*

To prove the theorem we prepare several lemmas. Let $P = P(v,d)$ be a stacked d-polytope with the vertex set V, $\sharp(V) = v$, $\Delta = \Delta(P)$ the boundary complex of P, and $\mathcal{F}$ a facet of P with the vertex set X. Let P' denote a stacked d-polytope with $(v+1)$-vertices which is obtained by building a shallow pyramid over $\mathcal{F}$ with a new vertex α, and Δ' the boundary complex of P'. Let $V' = V \bigcup \{\alpha\}$ be the vertex set of Δ' and W a subset of V'. We fix a base field k.

(4.4) LEMMA. (a) *If* $\alpha \notin W$ *and* $X \not\subset W$, *then*

$$\Delta'_W = \Delta_W.$$

(b) *If* $\alpha \notin W$, $W \neq V$ *and* $X \subset W$, *then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_W; k) & (i \neq d-2) \\ \dim_k \tilde{H}_i(\Delta_W; k) + 1 & (i = d-2). \end{cases}$$

(c) *If* $\alpha \in W$ *and* $X \bigcap W \neq \emptyset$, *then, for each* i, *we have*

$$\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W-\{\alpha\}}; k).$$

(d) *If* $\alpha \in W$, $W \neq \{\alpha\}$ *and* $X \bigcap W = \emptyset$, *then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k) & (i \neq 0) \\ \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k) + 1 & (i = 0). \end{cases}$$

Proof. (a) In general, $\Delta' = (\Delta - \{X\}) \bigcup \{\sigma \subset V' \mid \sigma \subset X \bigcup \{\alpha\}, \sigma \neq X\}$. Hence, we have $\Delta'_W = \Delta_W$ if $\alpha \notin W$ and $X \not\subset W$.

(b) Let Γ denote the set of all subsets of X and set $\partial\Gamma = \Gamma - \{X\}$. Then $\Delta'_W \bigcup \Gamma = \Delta_W$ and $\Delta'_W \bigcap \Gamma = \partial\Gamma$. Since Γ is a simplicial $(d-1)$-ball, $\partial\Gamma$ is a simplicial $(d-2)$-sphere and $\tilde{H}_{d-1}(\Delta_W; k) = 0$, the required equalities follow from the reduced Mayer–Vietoris exact sequence

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_i(\partial\Gamma; k) \longrightarrow \tilde{H}_i(\Gamma; k) \bigoplus \tilde{H}_i(\Delta'_W; k) \longrightarrow \tilde{H}_i(\Delta_W; k) \\ \longrightarrow \tilde{H}_{i-1}(\partial\Gamma; k) \longrightarrow \tilde{H}_{i-1}(\Gamma; k) \bigoplus \tilde{H}_{i-1}(\Delta'_W; k) \longrightarrow \tilde{H}_{i-1}(\Delta_W; k) \\ \longrightarrow \cdots \end{aligned}$$

(c) If $X \subset W$, then the geometric realization of Δ'_W is homeomorphic to that of $\Delta_{W-\{\alpha\}}$. Thus $\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W-\{\alpha\}}; k)$ for each i. On the other hand, if $X \bigcap W \neq X$, then $\Delta_{W-\{\alpha\}} \bigcup \Delta'_{W\cap(\{\alpha\}\cup X)} = \Delta'_W$ and $\Delta_{W-\{\alpha\}} \bigcap \Delta'_{W\cap(\{\alpha\}\cup X)} = \Delta_{W\cap X}$. Since both $\Delta'_{W\cap(\{\alpha\}\cup X)}$ and $\Delta_{W\cap X}$ are contractible, again the reduced Mayer–Vietoris exact sequence guarantees the desired equalities.

(d) Since Δ'_W is the disjoint union of $\Delta_{W-\{\alpha\}}$ and one point $\{\alpha\}$, we immediately have the required equalities. Q. E. D.

(4.5) LEMMA. *Let* $\Delta = \Delta(P)$ *denote the boundary complex of a stacked d-polytope* $P = P(v, d)$ *with the vertex set* V, $\sharp(V) = v$. *Then, for every non-empty subset* W *of* V *with* $W \neq V$ *and for each* $i \neq 0, d-2$, *we have*

$$\tilde{H}_i(\Delta_W; k) = 0.$$

Proof. If $v = d + 1$, i.e., P is a d-simplex, then Δ_W is contractible. Hence, $\tilde{H}_i(\Delta_W; k) = 0$ for each i. We now work with the same situation as in the above Lemma 4.4 and suppose that $\tilde{H}_i(\Delta_W; k) = 0$ for every non-empty subset W of V with $W \neq V$ and for each $i \neq 0, d-2$. Let W be a non-empty subset of V' with $W \neq V'$. If $W = V' - \{\alpha\}$, then Δ'_W is a simplicial $(d-1)$-ball. Hence, $\tilde{H}_i(\Delta_W; k) = 0$ for each i. Moreover, if $W = \{\alpha\}$, then $\tilde{H}_i(\Delta_W; k) = 0$ for each i. On the other hand, if W is a non-empty subset of V' with $W \neq V'$ such that $W \neq V$ and $W \neq \{\alpha\}$, and if $i \neq 0, d-2$, then $\dim_k \tilde{H}_i(\Delta'_W; k) = \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k)$ by Lemma 4.4. Hence, $\tilde{H}_i(\Delta_W; k) = 0$ as desired. Q. E. D.

Fix $d \geq 3$, and keep the notation P, P', Δ and Δ' as above. Let β_{i_j} be the i_j-th Betti number of $k[\Delta]$ and β'_{i_j} the i_j-th Betti number of $k[\Delta']$.

(4.6) LEMMA. *For each $i \geq 1$ we have*

$$\beta'_{i_{i+1}} = \beta_{i_{i+1}} + \beta_{i-1_i} + \binom{v-d}{i}.$$

Proof. By virtue of Hochster's formula as well as Lemma 4.4, we have

$$\begin{aligned}
\beta'_{i_{i+1}} &= \sum_{W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\
&= \sum_{\alpha \notin W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) + \sum_{\alpha \in W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\
&= \sum_{W \subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W; k) + \sum_{W \subset V,\ \sharp(W)=i} \dim_k \tilde{H}_0(\Delta_W; k) + \binom{v-d}{i} \\
&= \beta_{i_{i+1}} + \beta_{i-1_i} + \binom{v-d}{i}
\end{aligned}$$

as desired. Q. E. D.

(4.7) LEMMA. *Let $\Delta = \Delta(P)$ denote the boundary complex of a stacked d-polytope $P = P(v,d)$, $d \geq 3$, with v vertices. Then, for each $1 \leq i \leq v-d-1$, the i_{i+1}-th Betti number of $k[\Delta] = A/I_\Delta$ over A is*

$$\beta^A_{i_{i+1}}(k[\Delta]) = i\binom{v-d}{i+1}.$$

Proof. Thanks to Lemma 4.6, we have

$$\begin{aligned}
\beta^A_{i_{i+1}}(k[\Delta]) &= i\binom{v-d-1}{i+1} + (i-1)\binom{v-d-1}{i} + \binom{v-d-1}{i} \\
&= i(\binom{v-d-1}{i+1} + \binom{v-d-1}{i}) \\
&= i\binom{v-d}{i+1}
\end{aligned}$$

as required. Q. E. D.

Proof of Theorem 4.1. We are now in the position to give a proof of Theorem 4.1. Since $\Delta = \Delta(P(v,d))$ is a simplicial $(d-1)$-sphere with v vertices, we know that the homological dimension of $k[\Delta] = A/I_\Delta$ over A is $\mathrm{hd}_A(k[\Delta]) = v - d$ and that $\beta^A_{i_j}(k[\Delta]) = \beta^A_{v-d-i_{v-j}}(k[\Delta])$ for every i and j. By Lemma 4.5, we have $\beta^A_{i_j}(k[\Delta]) = 0$ for each $1 \le i \le v-d-1$ and for each $j \neq i+1, i+d-1$. On the other hand, Lemma 4.7 enables us to compute $b_i = \beta^A_{i_{i+1}}(k[\Delta]) = \beta^A_{v-d-i_{v-i-1}}(k[\Delta])$ for each $1 \le i \le v-d-1$. Hence, we obtain a desired minimal free resolution of $k[\Delta]$ over A.

§5. Connectivity of comparability graghs of distributive lattices

We now study the comparability graphs of finite distributive lattices. Every partially ordered set ("poset" for short) is finite. A *poset ideal* in a poset P is a subset $I \subset P$ such that $\alpha \in I$, $\beta \in P$ and $\beta \le \alpha$ together imply $\beta \in I$. A *clutter* is a poset in which no two elements are comparable. A *chain* of a poset P is a totally ordered subset of P. The *length* of a chain C is $\ell(C) := \sharp(C) - 1$. The *rank* of a poset P is defined to be $\mathrm{rank}(P) := \max\{\ell(C)\ ;\ C \text{ is a chain of } P\}$. Given a poset P, we write $\Delta(P)$ for the set of all chains of P. Then $\Delta(P)$ is a simplicial complex on the vertex set P, which is called the *order complex* of P. The *comparability graph* $\mathrm{Com}(P)$ of a poset P is the 1-skeleton $\Delta^{(1)}(P)$ of the order complex $\Delta(P)$. When $x \le y$ in a poset P, we define the closed interval $[x,y]$ to be the subposet $\{z \in P\ ;\ x \le z \le y\}$ of P.

A *lattice* is a poset L such that any two elements α and β of L have a greatest lower bound $\alpha \wedge \beta$ and a least upper bound $\alpha \vee \beta$. Let $\hat{0}$ (resp. $\hat{1}$) denote the unique minimal (resp. maximal) element of a lattice L. A lattice L is called *distributive* if the equalities $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and $\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ hold for all $\alpha, \beta, \gamma \in L$. Every closed interval of a distributive lattice is again a distributive lattice. A fundamental structure theorem for (finite) distributive lattices (see, e.g., [Sta$_2$, p. 106]) guarantees that, for every finite distributive lattice L, there exists a unique poset P such that $L = J(P)$, where $J(P)$ is the poset which consists of all poset ideals of P, ordered by inclusion. We say that a distributive lattice $L = J(P)$ is *planar* if P contains no three-element clutter. A *boolean lattice* is a distributive lattice $L = J(P)$ such that P is a clutter.

A chain $\mathcal{C} : \hat{0} = \alpha_0 < \alpha_1 < \cdots < \alpha_{s-1} < \alpha_s = \hat{1}$ of a distributive lattice L is called *essential* if each closed interval $[\alpha_i, \alpha_{i+1}]$ is a boolean lattice. In particular, all maximal chains of L is essential. Moreover, the chain $\hat{0} < \hat{1}$ of L is essential if and only if L is a boolean lattice. An essential chain $\mathcal{C} : \hat{0} = \alpha_0 < \alpha_1 < \cdots < \alpha_{s-1} < \alpha_s = \hat{1}$ is called *fundamental* if, for each $1 \leq i < s$, the subchain $\mathcal{C} - \{\alpha_i\}$ is not essential.

(5.1) LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = v$ and i an integer with $1 \leq i < v$. Then the 1-skeleton $\Delta^{(1)}$ of Δ is i-connected if and only if $\beta_{v-i,v-i+1}(k[\Delta]) = 0$.*

Proof. The 1-skeleton $\Delta^{(1)}$ is i-connected if and only if, for every subset W of V with $\sharp(W) = i - 1$, we have $\tilde{H}_0(\Delta^{(1)}_{V-W}; k)$ $(= \tilde{H}_0(\Delta_{V-W}; k)) = 0$. Moreover, by virtue of Eq. (2), $\tilde{H}_0(\Delta_{V-W}; k) = 0$ for every subset W of V with $\sharp(W) = i - 1$ if and only if $\beta_{v-i,v-i+1}(k[\Delta]) = 0$ as desired. Q. E. D.

The following Lemma 5.2 is discussed in [H_2].

(5.2) LEMMA ([H_2]). *Let L be a ditributive lattice of rank $d - 1$ with $\sharp(L) = v$ and $\Delta = \Delta(L)$ its order complex. Then the $(v - d, v - d + i)$-th Betti number $\beta_{v-d,v-d+i}(k[\Delta])$ is equal to the number of fundamental chains of L of length $d - i - 1$.*

We are now in the position to give the main result of the this chapter.

(3.3) THEOREM. *Suppose that a finite distributive lattice L of rank $d - 1$ is non-planar. Then the comparability graph* Com(L) *of L is d-connected.*

Proof. Let $P = \{p_1, p_2, \cdots, p_{d-1}\}$ denote a poset with $L = J(P)$ and $\mathcal{M} : \hat{0} = \alpha_0 < \alpha_1 < \cdots < \alpha_{d-2} < \alpha_{d-1} = \hat{1}$ an arbitrary maximal chain of L. We may assume that each α_i is the poset ideal $\{p_1, p_2, \cdots, p_i\}$ of P. Since L is non-planar, there exists a three-element clutter, say, $\{p_l, p_m, p_n\}$ with $1 \leq l < m < n \leq d - 1$. Hence, for some $l \leq i < m$, p_i and p_{i+1} are incomparable in P, and for some $m \leq j < n$, p_j and p_{j+1} are incomparable in P. Let $l \leq i < m$ (resp. $m \leq j < n$) denote the least (resp. greatest) integer i (resp. j) with the above property. Then $\beta = \{p_1, \cdots, p_{i-1}, p_{i+1}\}$ and $\gamma = \{p_1, \cdots, p_{j-1}, p_{j+1}\}$ both are poset ideals of P. Moreover, $\alpha_{i-1} < \beta < \alpha_{i+1}$ in L with $\beta \neq \alpha_i$ and $\alpha_{j-1} < \gamma < \alpha_{j+1}$ in L with $\gamma \neq \alpha_j$. Thus the closed intervals $[\alpha_{i-1}, \alpha_{i+1}]$ and $[\alpha_{j-1}, \alpha_{j+1}]$ both are boolean. Hence, if $i + 1 \leq j - 1$, then the chain $\mathcal{M} - \{\alpha_i, \alpha_j\}$ is essential. On the other hand, if $i + 1 > j - 1$, i.e., $i = m - 1$ and $j = m$, then $p_l < p_{l+1} < \cdots < p_{m-1}$

and $p_{m+1} < p_{m+2} < \cdots < p_n$ in P; thus $\{p_{m-1}, p_m, p_{m+1}\}$ is a clutter of P. Hence the closed interval $[\alpha_{m-2}, \alpha_{m+1}]$ of L is boolean, and the chain $\mathcal{M} - \{\alpha_{m-1}, \alpha_m\}$ is essential. Consequently, there exists no fundamental chain of L of length $d-2$. Thus, by Lemma 5.2, $\beta_{v-d,v-d+1}(k[\Delta(L)]) = 0$. Hence, by Lemma 5.1 again, the comparability graph $\mathrm{Com}(L) = \Delta^{(1)}(L)$ of L is d-connected as desired. Q. E. D.

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